

9. cvičení - řešení

Příklad 1 (a)

$$\begin{aligned}\frac{2x+1}{(x+1)(2x+3)} &= \frac{A}{x+1} + \frac{B}{2x+3} \\ 2x+1 &= A(2x+3) + B(x+1) \\ x = -1 &\implies -1 = A \cdot (3-2) \implies A = -1 \\ x = -\frac{3}{2} &\implies 1-3 = B \cdot \frac{-1}{2} \implies B = 4\end{aligned}$$

$$\int \frac{2x+1}{(x+1)(2x+3)} dx \stackrel{\text{lin.}}{=} -\int \frac{1}{x+1} dx + 4 \int \frac{1}{2x+3} dx \stackrel{c}{=} -\log|x+1| + 2 \log|2x+3|$$

Příklad 1 (b)

$$\begin{aligned}\frac{x-1}{x(x+1)^3} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3} \\ x-1 &= A(x+1)^3 + Bx(x+1)^2 + Cx(x+1) + Dx \\ x=0 &\implies -1 = A \cdot 1^3 \implies A = -1 \\ x=-1 &\implies -2 = -D \implies D = 2 \\ x-1 &= -(x^3+3x^2+3x+1) + B(x^3+2x^2+x) + C(x^2+x) + 2x \\ x-1 &= -x^3-3x^2-3x-1 + Bx^3+2Bx^2+Bx+Cx^2+Cx+2x \\ x-1 &= (B-1)x^3 + (-3+2B+C)x^2 + (-1+B+C)x-1\end{aligned}$$

Porovnáním koeficientů u jednotlivých mocnin x dostáváme soustavu:

$$\begin{aligned}0 &= B-1 \implies B=1 \\ 0 &= C+2B-3 \\ 1 &= C+B-1 \stackrel{1}{\implies} 1 = C+1-1 \implies C=1\end{aligned}$$

Lze snadno ověřit, že druhá rovnice je též splněna: $1+2 \cdot 1-3=0$.

$$\begin{aligned}\int \frac{x-1}{x(x+1)^3} dx &\stackrel{\text{lin.}}{=} -\int \frac{1}{x} dx + \int \frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{(x+1)^3} dx = \\ &= |y=x+1, dy=dx| = -\log|x| + \int \frac{1}{y} + \frac{1}{y^2} + \frac{2}{y^3} dy = \\ &\stackrel{c}{=} -\log|x| + \log|y| - \frac{1}{y} - \frac{1}{y^2} = -\log|x| + \log|x+1| - \frac{1}{(x+1)} - \frac{1}{(x+1)^2}\end{aligned}$$

Příklad 1 (c)

$$\frac{x^{15} - 3}{x - 1} = x^{14} + x^{13} + \dots + x + 1 - \frac{2}{x - 1} \quad \text{dle dělení mnohočlenu mnohočlenem}$$

$$\int \frac{x^{15} - 3}{x - 1} dx \stackrel{\text{lin.}}{=} \int x^{14} + x^{13} + \dots + x + 1 dx - \int \frac{2}{x - 1} dx \stackrel{c}{=} \frac{x^{15}}{15} + \frac{x^{14}}{14} + \dots + \frac{x^2}{2} + x - 2 \log |x - 1|$$

Příklad 1 (d)

$$\begin{aligned} \frac{x^4}{x^4 + 5x^2 + 4} &= \frac{x^4 + 5x^2 + 4 - 5x^2 - 4}{x^4 + 5x^2 + 4} = 1 - \frac{5x^2 + 4}{x^4 + 5x^2 + 4} \\ x^4 + 5x^2 + 4 &= (x^2 + 1)(x^2 + 4) \end{aligned}$$

$$\begin{aligned} \frac{-5x^2 - 4}{(x^2 + 1)(x^2 + 4)} &= \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 4} \\ -5x^2 - 4 &= Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D \\ -5x^2 - 4 &= (A + C)x^3 + (B + D)x^2 + (4A + C)x + 4B + D \end{aligned}$$

Porovnáním koeficientů u jednotlivých mocnin x dostáváme soustavu:

$$\begin{aligned} 0 &= A + C \stackrel{4}{\Rightarrow} 0 = -3A \Rightarrow A = 0, C = 0 \\ -5 &= B + D \\ 0 &= 4A + C \\ -4 &= 4B + D \stackrel{2}{\Rightarrow} -4 = 4B - 5 - B \Rightarrow 3B = 1 \Rightarrow B = \frac{1}{3}, D = -\frac{16}{3} \end{aligned}$$

$$\begin{aligned} \int \frac{x^4}{x^4 + 5x^2 + 4} dx &\stackrel{\text{lin.}}{=} \int 1 dx + \frac{1}{3} \int \frac{1}{x^2 + 1} dx - \frac{16}{3} \int \frac{1}{x^2 + 4} dx = \\ &\stackrel{\text{lin.}}{=} x + \frac{1}{3} \arctan x - \frac{16}{3} \cdot \frac{1}{4} \int \frac{1}{\left(\frac{x}{2}\right)^2 + 1} dx = \left| y = \frac{x}{2}, dy = \frac{1}{2} dx \right| = x + \frac{1}{3} \arctan x - \frac{16}{6} \int \frac{1}{y^2 + 1} dy = \\ &\stackrel{c}{=} x + \frac{1}{3} \arctan x - \frac{16}{6} \arctan y = x + \frac{1}{3} \arctan x - \frac{8}{3} \arctan \frac{x}{2} \end{aligned}$$

Příklad 1 (e)

$$\begin{aligned} \frac{x^4}{x^4 + 2x^2 - 3} &= \frac{x^4 + 2x^2 - 3}{x^4 + 2x^2 - 3} + \frac{-2x^2 + 3}{x^4 + 2x^2 - 3} = 1 + \frac{-2x^2 + 3}{x^4 + 2x^2 - 3} \\ x^4 + 2x^2 - 3 &= (x^2 - 1)(x^2 + 3) = (x - 1)(x + 1)(x^2 + 3) \end{aligned}$$

$$\begin{aligned}
\frac{-2x^2 + 3}{(x-1)(x+1)(x^2+3)} &= \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+3} \\
-2x^2 + 3 &= A(x-1)(x^2+3) + B(x+1)(x^2+3) + (Cx+D)(x^2-1) \\
x=1 &\implies 3-2 = B \cdot 2 \cdot 4 \implies B = \frac{1}{8} \\
x=-1 &\implies 1 = A \cdot (-2) \cdot 4 \implies A = -\frac{1}{8} \\
-2x^2 + 3 &= -\frac{1}{8}x^3 - \frac{3}{8}x + \frac{1}{8}x^2 + \frac{3}{8} + \frac{1}{8}x^3 + \frac{3}{8}x + \frac{1}{8}x^2 + \frac{3}{8} + Cx^3 - Cx + Dx^2 - D \\
-2x^2 + 3 &= Cx^3 + \left(\frac{1}{4} + D\right)x^2 - Cx + \frac{6}{8} - D
\end{aligned}$$

Porovnáním koeficientů u jednotlivých mocnin x dostáváme soustavu:

$$\begin{aligned}
C &= 0 \\
-2 &= D + \frac{1}{4} \implies D = -2 - \frac{1}{4} = -\frac{9}{4} \\
0 &= -C \\
3 &= \frac{6}{8} - D
\end{aligned}$$

Zřejmě dostáváme $C = 0, D = -\frac{9}{4}$, což splňuje všechny uvedené rovnice.

Máme tedy: $A = -\frac{1}{8}, B = \frac{1}{8}, C = 0, D = -\frac{9}{4}$

$$\begin{aligned}
\int \frac{x^4}{x^4 + 2x^2 - 3} dx &\stackrel{\text{lin.}}{=} \int 1 dx - \frac{1}{8} \int \frac{1}{x+1} dx + \frac{1}{8} \int \frac{1}{x-1} dx - \frac{9}{4} \int \frac{1}{x^2+3} dx = \\
&= |y = x+1, dy = dx, z = x-1, dz = dx| = x - \frac{1}{8} \int \frac{1}{y} dy + \frac{1}{8} \int \frac{1}{z} dz - \frac{9}{4} \cdot \frac{1}{3} \int \frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2 + 1} dx = \\
&= \left| u = \frac{x}{\sqrt{3}}, du = \frac{1}{\sqrt{3}} dx \right| \stackrel{\text{lin.}}{=} x - \frac{1}{8} \log |y| + \frac{1}{8} \log |z| - \frac{3\sqrt{3}}{4} \int \frac{1}{1+u^2} du = \\
&\stackrel{c}{=} x - \frac{1}{8} \log |y| + \frac{1}{8} \log |z| - \frac{3\sqrt{3}}{4} \arctan u = x - \frac{1}{8} \log |x+1| + \frac{1}{8} \log |x-1| - \frac{3\sqrt{3}}{4} \arctan \frac{x}{\sqrt{3}}
\end{aligned}$$

Příklad 1 (f)

$$\begin{aligned}
\frac{1}{(x^2-4x+4)(x^2-4x+5)} &= \frac{1}{(x-2)^2(x^2+4x+5)} \\
\frac{1}{(x-2)^2(x^2+4x+5)} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2-4x+5} \\
1 &= A(x-2)(x^2-4x+5) + B(x^2-4x+5) + (Cx+D)(x^2-4x+4) \\
x=2 &\implies 1 = B(4-8+5) \implies B = 1 \\
1 &= Ax^3 - 4Ax^2 + 5Ax - 2Ax^2 + 8Ax - 10A + x^2 - 4x + 5 + Cx^3 - 4Cx^2 + 4Cx + Dx^2 - 4Dx + 4D \\
1 &= (A+C)x^3 + (-4A-2A+1-4C+D)x^2 + (5A+8A-4+4C-4D)x - 10A+5+4D
\end{aligned}$$

Porovnáním koeficientů u jednotlivých mocnin x dostáváme soustavu:

$$0 = A + C \implies C = -A \xrightarrow{3} A = 0$$

$$0 = -6A + 1 - 4C + D \xrightarrow{1} -1 = 2C + D \implies D = -1 - 2C \xrightarrow{3} D = -1$$

$$0 = 13A - 4 + 4C - 4D \xrightarrow{1} 4 = -9C - 4D \xrightarrow{2} 4 = -9C + 4 + 8C \implies C = 0$$

$$1 = -10A + 5 + 4D \xrightarrow{1} -5 = 10C + 4D$$

Dostáváme: $A = 0, B = 1, C = 0, D = -1$.

$$\begin{aligned} \int \frac{1}{(x^2 - 4x + 4)(x^2 - 4x + 5)} dx &\stackrel{\text{lin.}}{=} \int \frac{1}{(x-2)^2} dx - \int \frac{1}{x^2 - 4x + 5} dx = \\ &= \int \frac{1}{(x-2)^2} dy - \int \frac{1}{(x-2)^2 + 1} dx = |y = x - 2, dy = dx| = \int \frac{1}{y^2} dy - \int \frac{1}{y^2 + 1} dy = \\ &\stackrel{c}{=} \frac{-1}{y} - \arctan y = -\frac{1}{x-2} - \arctan(x-2) \end{aligned}$$

Příklad 1 (g)

$$\begin{aligned} \frac{x^2}{(x^2 + 2x + 2)^2} &= \frac{Ax + B}{x^2 + 2x + 2} + \frac{Cx + D}{(x^2 + 2x + 2)^2} \\ x^2 &= (Ax + B)(x^2 + 2x + 2) + Cx + D \\ x^2 &= Ax^3 + 2Ax^2 + 2Ax + Bx^2 + 2Bx + 2B + Cx + D \\ x^2 &= Ax^3 + (2A + B)x^2 + (2A + 2B + C)x + 2B + D \end{aligned}$$

Porovnáním koeficientů u jednotlivých mocnin x dostáváme soustavu:

$$0 = A$$

$$1 = 2A + B \xrightarrow{1} B = 1$$

$$0 = 2A + 2B + C \xrightarrow{1+2} C = -2$$

$$0 = 2B + D \xrightarrow{2} D = -2$$

$$\begin{aligned} \int \frac{x^2}{(x^2 + 2x + 2)^2} dx &\stackrel{\text{lin.}}{=} \int \frac{1}{x^2 + 2x + 2} dx - 2 \int \frac{x+1}{(x^2 + 2x + 2)^2} dx = \\ &\stackrel{\text{lin.}}{=} \int \frac{1}{(x+1)^2 + 1} dx - \int \frac{2x+2}{(x^2 + 2x + 2)^2} dx = \\ &= |y = x + 1, dy = dx, z = x^2 + 2x + 2, dz = (2x + 2)dx| = \\ &= \int \frac{1}{y^2 + 1} dy - \int \frac{1}{z^2} dz = \arctan y + \frac{1}{z} = \\ &= \arctan(x+1) + \frac{1}{x^2 + 2x + 2} \end{aligned}$$

Níže je uvedeno jen částečné řešení

rozklad na parciální zlomky je algoritmický a snadno jej provedete sami

Příklad 1 (h)

$$\begin{aligned}
 \int \frac{x+1}{x^2-x+1} dx &= |y = x^2 - x + 1, dy = (2x-1) dx| = \int \frac{2(x+1)}{2(x^2-x+1)} dx = \\
 &= \int \frac{2x+2-3+3}{2(x^2-x+1)} dx \stackrel{\text{lin.}}{=} \frac{1}{2} \int \frac{2x-1}{x^2-x+1} dx + \frac{3}{2} \int \frac{1}{x^2-x+1} dx = \\
 &= \frac{1}{2} \int \frac{1}{y} dy + \frac{3}{2} \int \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}} dx = \frac{1}{2} \log |y| + \frac{3}{2} \cdot \frac{4}{3} \int \frac{1}{\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2 + 1} dx = \\
 &= \left| z = \sqrt{\frac{4}{3}} \left(x - \frac{1}{2}\right), dz = \sqrt{\frac{4}{3}} \right| = \\
 &= \frac{1}{2} \log |x^2 - x + 1| + 2\sqrt{\frac{3}{4}} \int \frac{1}{z^2 + 1} dz \stackrel{c}{=} \frac{1}{2} \log |x^2 - x + 1| + \sqrt{3} \arctan z = \\
 &= \frac{1}{2} \log |x^2 - x + 1| + \sqrt{3} \arctan \left(\sqrt{\frac{4}{3}} \left(x - \frac{1}{2}\right) \right) = \\
 &= \frac{1}{2} \log |x^2 - x + 1| + \sqrt{3} \arctan \left(\frac{2}{\sqrt{3}} \cdot \frac{2x-1}{2} \right)
 \end{aligned}$$

$$\int \frac{1}{x+1} dx = |y = x+1, dy = dx| = \int \frac{1}{y} dy \stackrel{c}{=} \log |y| = \log |x+1|$$

Příklad 1 (i)

$$\begin{aligned}
 \int \frac{-3x+2}{x^2-x+2} dx &= |y = x^2 - x + 2, dy = 2x-1 dx| = \int \frac{-\frac{2}{3}}{-\frac{2}{3}} \cdot \frac{-3x+2}{x^2-x+2} dx = \\
 &= -\frac{3}{2} \int \frac{2x-\frac{4}{3}+1-1}{x^2-x+2} dx \stackrel{\text{lin.}}{=} -\frac{3}{2} \int \frac{2x-1}{x^2-x+2} dx - \frac{3}{2} \cdot \frac{-1}{3} \int \frac{1}{x^2-x+2} dx = \\
 &= -\frac{3}{2} \int \frac{1}{y} dy + \frac{1}{2} \int \frac{1}{(x-\frac{1}{2})^2 + \frac{7}{4}} dx = -\frac{3}{2} \log |y| + \frac{1}{2} \cdot \frac{4}{7} \int \frac{1}{\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{7}{4}}}\right)^2 + 1} dx = \\
 &= \left| z = \sqrt{\frac{4}{7}} \left(x - \frac{1}{2}\right), dz = \sqrt{\frac{4}{7}} dx \right| = -\frac{3}{2} \log |x^2 - x + 2| + \frac{2}{7} \cdot \sqrt{\frac{7}{4}} \int \frac{1}{z^2 + 1} dz = \\
 &\stackrel{c}{=} -\frac{3}{2} \log |x^2 - x + 2| + \frac{\sqrt{7}}{7} \arctan z = -\frac{3}{2} \log |x^2 - x + 2| + \frac{1}{\sqrt{7}} \arctan \left(\sqrt{\frac{4}{7}} \left(x - \frac{1}{2}\right) \right) = \\
 &= -\frac{3}{2} \log |x^2 - x + 2| + \frac{1}{\sqrt{7}} \arctan \left(\frac{2}{\sqrt{7}} \cdot \frac{2x-1}{2} \right) = -\frac{3}{2} \log |x^2 - x + 2| + \frac{1}{\sqrt{7}} \arctan \frac{2x-1}{\sqrt{7}}
 \end{aligned}$$

Příklad 1 (j)

$$\begin{aligned}
\int \frac{x+1}{x^2+x+1} dx &= |y = x^2 + x + 1, dy = (2x+1)dx| = \int \frac{2}{2} \cdot \frac{x+1}{x^2+x+1} dx = \\
&\stackrel{\text{lin.}}{=} \frac{1}{2} \int \frac{2x+2-1+1}{x^2+x+1} dx \stackrel{\text{lin.}}{=} \frac{1}{2} \int \frac{2x+1}{x^2+x+1} dx + \frac{1}{2} \int \frac{1}{x^2+x+1} dx = \\
&= \frac{1}{2} \int \frac{1}{y} dy + \frac{1}{2} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx = \\
&= \frac{1}{2} \log |y| + \frac{1}{2} \cdot \frac{4}{3} \int \frac{1}{\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2 + 1} dx = \left| z = \sqrt{\frac{4}{3}} \left(x + \frac{1}{2}\right), dz = \sqrt{\frac{4}{3}} dx \right| = \\
&= \frac{1}{2} \log |x^2 + x + 1| + \frac{2}{3} \sqrt{\frac{3}{4}} \int \frac{1}{z^2 + 1} dz \stackrel{c}{=} \frac{1}{2} \log |x^2 + x + 1| + \frac{\sqrt{3}}{3} \arctan z = \\
&= \frac{1}{2} \log |x^2 + x + 1| + \frac{1}{\sqrt{3}} \arctan \sqrt{\frac{4}{3}} \left(x + \frac{1}{2}\right) = \\
&= \frac{1}{2} \log |x^2 + x + 1| + \frac{1}{\sqrt{3}} \arctan \frac{2}{\sqrt{3}} \frac{2x+1}{2} = \frac{1}{2} \log |x^2 + x + 1| + \frac{1}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
\int \frac{x-1}{x^2-x+1} dx &= |y = x^2 - x + 1, dy = (2x-1)dx| \stackrel{\text{lin.}}{=} \frac{1}{2} \int \frac{2x-2+1-1}{x^2-x+1} dx = \\
&\stackrel{\text{lin.}}{=} \frac{1}{2} \int \frac{2x-1}{x^2-x+1} dx - \frac{1}{2} \int \frac{1}{x^2-x+1} dx = \frac{1}{2} \int \frac{1}{y} dy - \frac{1}{2} \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} dx = \\
&= \frac{1}{2} \log |y| - \frac{1}{2} \cdot \frac{4}{3} \int \frac{1}{\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2 + 1} dx = \left| z = \sqrt{\frac{4}{3}} \left(x - \frac{1}{2}\right), dz = \sqrt{\frac{4}{3}} dx \right| = \\
&= \frac{1}{2} \log |x^2 - x + 1| - \frac{2}{3} \sqrt{\frac{3}{4}} \int \frac{1}{z^2 + 1} dz \stackrel{c}{=} \frac{1}{2} \log |x^2 - x + 1| - \frac{1}{\sqrt{3}} \arctan z = \\
&\stackrel{c}{=} \frac{1}{2} \log |x^2 - x + 1| - \frac{1}{\sqrt{3}} \arctan \sqrt{\frac{4}{3}} \left(x - \frac{1}{2}\right) = \\
&= \frac{1}{2} \log |x^2 - x + 1| - \frac{1}{\sqrt{3}} \arctan \frac{2}{\sqrt{3}} \frac{2x-1}{2} = \\
&= \frac{1}{2} \log |x^2 - x + 1| - \frac{1}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}}
\end{aligned}$$

Příklad 1 (k)

$$\int \frac{1}{2x+1} dx = |y = 2x+1, dy = 2dx| = \int \frac{1}{2y} dy \stackrel{c}{=} \frac{1}{2} \log |y| = \frac{1}{2} \log |2x+1|$$

$$\int \frac{2x+1}{x^2+x+1} dx = |y = x^2 + x + 1, dy = (2x+1)dx| = \int \frac{1}{y} dy \stackrel{c}{=} \log |y| = \log |x^2 + x + 1|$$

$$\begin{aligned}
\int \frac{7x+5}{(x^2+x+1)^2} dx &= \left| y = x^2 + x + 1, dy = (2x+1)dx \right| \stackrel{\text{lin.}}{=} 7 \int \frac{x + \frac{5}{7}}{(x^2+x+1)^2} dx = \\
&= \frac{7}{2} \int \frac{2x + \frac{10}{7} - \frac{3}{7} + \frac{3}{7}}{(x^2+x+1)^2} dx \stackrel{\text{lin.}}{=} \frac{7}{2} \int \frac{2x+1}{(x^2+x+1)^2} dx + \frac{7}{2} \cdot \frac{3}{7} \int \frac{1}{(x^2+x+1)^2} dx = \\
&= \frac{7}{2} \int \frac{1}{y^2} dy + \frac{3}{2} \int \frac{1}{(x^2+x+1)^2} dx = -\frac{7}{2y} + \frac{3}{2} \int \frac{1}{(x^2+x+1)^2} dx = \\
&= -\frac{7}{2(x^2+x+1)} + \frac{3}{2} \int \frac{1}{(x^2+x+1)^2} dx
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{(x^2+x+1)^2} dx &= \int \frac{1}{\left((x+\frac{1}{2})^2 + \frac{3}{4}\right)^2} dx = \frac{16}{9} \int \frac{1}{\left(\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2 + 1\right)^2} dx = \\
&= \left| u = \sqrt{\frac{4}{3}} \left(x + \frac{1}{2}\right), du = \sqrt{\frac{4}{3}} dx \right| = \frac{16}{9} \cdot \frac{\sqrt{3}}{2} \int \frac{1}{(u^2+1)^2} du = \\
&= \left| u = \tan t, du = \frac{1}{\cos^2 t} dt \right| = \frac{8\sqrt{3}}{9} \int \frac{1}{(\tan^2 t + 1)^2} \frac{1}{\cos^2 t} dt = \\
&= \frac{8\sqrt{3}}{9} \int \frac{1}{\left(\frac{\sin^2 t + \cos^2 t}{\cos^2 t}\right)^2} \frac{1}{\cos^2 t} dt = \frac{8\sqrt{3}}{9} \int \cos^2 t dt = \frac{8\sqrt{3}}{9} \frac{1}{2} (\cos t \sin t + t)
\end{aligned}$$

Výše se využilo následujícího:

$$\begin{aligned}
\int \cos^2 x dx &= \int \cos x \cdot \cos x dx \stackrel{\text{PP}}{=} \sin x \cos x - \int \sin x (-\sin x) dx = \sin x \cos x + \int \sin^2 x dx = \\
&= \sin x \cos x + \int 1 - \cos^2 x dx \stackrel{\text{lin.}}{=} \sin x \cos x + x - \int \cos^2 x dx
\end{aligned}$$

Vyjádřením ze získané rovnosti dostáváme:

$$\int \cos^2 x dx = \frac{1}{2} (\sin x \cos x + x)$$

Přičemž: $u = \tan t$, tedy $t = \arctan t$. Navíc $1 + \tan^2 t = \frac{1}{\cos^2 t}$, tedy $\cos^2 t = \frac{1}{\tan^2 t + 1}$. Z toho plyne, že $\sin t \cos t = \frac{\sin t}{\cos t} \cos^2 t = \tan t \frac{1}{\tan^2 t + 1}$, tedy $\cos t \sin t = \frac{u}{u^2 + 1}$. Z toho plyne, že:

$$\int \frac{1}{(x^2+x+1)^2} dx \stackrel{c}{=} \frac{4\sqrt{3}}{9} \left(\arctan u + \frac{u}{u^2+1} \right) = \frac{4\sqrt{3}}{9} \left(\arctan \left(\frac{2x+1}{\sqrt{3}} \right) + \frac{\sqrt{\frac{4}{3}} \left(x + \frac{1}{2}\right)}{\left(\sqrt{\frac{4}{3}} \left(x + \frac{1}{2}\right)\right)^2 + 1} \right)$$

Příklad 1 (I)

Rovnice $x^4 + x^2 + 1 = 0$ je reciproká. Jak na reciproké rovnice vizte zde:

$$x^4 + x^2 + 1 = 0 \quad \text{vydělíme výrazem } x^2$$

$$x^2 + 1 + \frac{1}{x^2} = 0, \quad y := x + \frac{1}{x}, y^2 = \left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}$$

$$\implies y^2 - 1 = 0$$

$$(y + 1)(y - 1) = 0$$

Rovnice $x^4 + x^2 + 1 = 0$ lze substitucí $y = x + \frac{1}{x}$ přepsat na $(y + 1)(y - 1)$. Po provedení desubstituce dostáváme rozklad: $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$.

Rozklad na parciální zlomky získáme vyšetřením rovnosti:

$$\frac{x^2 + 1}{(x^4 + x^2 + 1)^2} = \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{(x^2 + x + 1)^2} + \frac{Ex + F}{x^2 - x + 1} + \frac{Gx + H}{(x^2 - x + 1)^2}$$

Dostaneme: $A = B = D = E = F = H = \frac{1}{4}, C = G = -\frac{1}{4}$.

$$\begin{aligned} \int \frac{x + 1}{x^2 + x + 1} dx &= \left| y = x^2 + x + 1, dy = (2x + 1)dx \right| = \frac{1}{2} \int \frac{2x + 2 - 1 + 1}{x^2 + x + 1} dx = \\ &\stackrel{\text{lin.}}{=} \frac{1}{2} \int \frac{2x + 1}{x^2 + x + 1} dx + \frac{1}{2} \int \frac{1}{x^2 + x + 1} dx = \frac{1}{2} \int \frac{1}{y} dy + \frac{1}{2} \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} dx = \\ &= \frac{1}{2} \log |y| + \frac{1}{2} \cdot \frac{4}{3} \int \frac{1}{\left(\frac{x + \frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2 + 1} dx = \left| z = \sqrt{\frac{4}{3}} \left(x + \frac{1}{2}\right), dz = \sqrt{\frac{4}{3}} dx \right| = \\ &= \frac{1}{2} \log |x^2 + x + 1| + \frac{2}{3} \cdot \frac{\sqrt{3}}{2} \int \frac{1}{z^2 + 1} dz \stackrel{c}{=} \frac{1}{2} \log |x^2 + x + 1| + \frac{1}{\sqrt{3}} \arctan z = \\ &= \frac{1}{2} \log |x^2 + x + 1| + \frac{1}{\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned}
\int \frac{x+1}{(x^2+x+1)^2} dx &= |y = x^2 + x + 1, dy = (2x+1)dx| = \frac{1}{2} \int \frac{2x+2-1+1}{(x^2+x+1)^2} dx = \\
&\stackrel{\text{lin.}}{=} \frac{1}{2} \int \frac{2x+1}{(x^2+x+1)^2} dx + \frac{1}{2} \int \frac{1}{\left((x+\frac{1}{2})^2 + \frac{3}{4}\right)^2} dx = \\
&= \frac{1}{2} \int \frac{1}{y^2} dy + \frac{1}{2} \int \frac{1}{\frac{\frac{9}{16} \left(1 + \left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2\right)^2} dx = \\
&= \left| u = \sqrt{\frac{4}{3}} \left(x + \frac{1}{2}\right), du = \sqrt{\frac{4}{3}} dx \right| = -\frac{1}{2y} + \frac{1}{2} \cdot \frac{16}{9} \cdot \frac{\sqrt{3}}{2} \int \frac{1}{(u^2+1)^2} du = \\
&= -\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9} \int \frac{1}{(u^2+1)^2} du = \left| u = \tan t, du = \frac{1}{\cos^2 t} dt \right| = \\
&= -\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9} \int \frac{1}{(\tan^2 t + 1)^2 \cos^2 t} dt = \\
&= -\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9} \int \frac{1}{\left(\frac{\sin^2 + \cos^2}{\cos^2}\right)^2 \cos^2 t} dt = \\
&\quad -\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9} \int \cos^2 t dt
\end{aligned}$$

Využijeme následujícího:

$$\begin{aligned}
\int \cos^2 x dx &= \int \cos x \cdot \cos x dx \stackrel{\text{PP}}{=} \sin x \cos x - \int \sin x (-\sin x) dx = \sin x \cos x + \int \sin^2 x dx = \\
&= \sin x \cos x + \int 1 - \cos^2 x dx \stackrel{\text{lin.}}{=} \sin x \cos x + x - \int \cos^2 x dx
\end{aligned}$$

Vyjádřením ze získané rovnosti dostáváme:

$$\int \cos^2 x dx = \frac{1}{2} (\sin x \cos x + x)$$

Pokračujeme ve výpočtu:

$$\begin{aligned}
\int \frac{x+1}{(x^2+x+1)^2} dx &= -\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9} \int \cos^2 t dt \stackrel{c}{=} -\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9} \frac{1}{2} (\sin t \cos t + t) = \\
&= -\frac{1}{2(x^2+x+1)} + \frac{2\sqrt{3}}{9} (\tan t \cos^2 t + t) = \\
&= \left| 1 + \tan^2 t = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \implies \cos^2 t = \frac{1}{\tan^2 t + 1} \right| = \\
&\stackrel{c}{=} -\frac{1}{2(x^2+x+1)} + \frac{2\sqrt{3}}{9} \left(\frac{u}{u^2+1} + \arctan u \right) = \\
&= -\frac{1}{2(x^2+x+1)} + \frac{2\sqrt{3}}{9} \left(\frac{\frac{2}{\sqrt{3}} \left(x + \frac{1}{2}\right)}{\left(\frac{2}{\sqrt{3}} \left(x + \frac{1}{2}\right)\right)^2 + 1} + \arctan \frac{2}{\sqrt{3}} \left(x + \frac{1}{2}\right) \right) = \\
&= -\frac{1}{2(x^2+x+1)} + \frac{2\sqrt{3}}{9} \left(\frac{\frac{2x+1}{\sqrt{3}}}{\left(\frac{2}{\sqrt{3}} \left(x + \frac{1}{2}\right)\right)^2 + 1} + \arctan \frac{2x+1}{\sqrt{3}} \right)
\end{aligned}$$

Zbytek analogicky.

Příklad 2, 3, 4

https://www.karlin.mff.cuni.cz/~kuncova/2324LS_MA2/11res.pdf